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LETTER TO THE EDITOR

The imbedding SO(4) meron solutions in the SU(N) Yang–Mills theory

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Abstract. The general forms of imbedding SO(4) meron solutions with vanishing total energy in the SU(N) Yang–Mills theory are obtained. The topological properties of the imbedding SO(4) instanton and meron solutions are discussed.

In a previous paper (Ma and Xu 1984) we discussed the rotation property of the SU(N) Yang–Mills gauge potential in the four-dimensional Euclidean space. By making use of the method of the phase factor of the standard differential loop (Gu 1981, Ma 1984), it is proved that the spherically symmetric gauge potential must be synchrospherically symmetric, i.e., there is a gauge (central gauge) in which the potentials before and after four-dimensional rotation are related by a global gauge transformation which is an N-dimensional representation $\mathscr{D}$ of the SO(4) group,

$$R_{\mu\nu}W_\nu(R^{-1}x)=\mathscr{D}(R^{-1})W_\mu(x)\mathscr{D}(R). \tag{1}$$

If $W_\mu(x)$ is composed of the generators $I_{\mu\nu}$ of $\mathscr{D}$, the general form of $W_\mu(x)$ is

$$W_\mu(x)=\phi_1(x^2)I_{\mu\nu}x_\nu+\phi_2(x^2)\tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}I_{\nu\rho}x_\lambda. \tag{2}$$

Obviously, this W_μ is the imbedding SO(4) potential. Inserting equation (2) into the sourceless Yang–Mills equation, we have (Ma and Xu 1984)

$$2x^2(\ddot\phi_1\pm\ddot\phi_2)+6(\dot\phi_1\pm\dot\phi_2)-e^2x^2(\phi_1\pm\phi_2)^3+3e(\phi_1\pm\phi_2)^2=0, \tag{3}$$

where dots refer to differentiation with respect to x^2. In addition to the instanton solutions we obtained from equation (3) (Ma and Xu 1984)

$$\begin{aligned}&\phi_1+\phi_2=2/e(x^2+a^2), \qquad \phi_1-\phi_2=2/e(x^2+b^2),\\ &a^2>0, \qquad b^2>0,\end{aligned} \tag{4}$$

we have the meron solutions

$$\phi_1\pm\phi_2=M_\pm/ex^2, \qquad M_\pm=0,\ 1,\ 2, \tag{5}$$

which are singular at the origin. The solutions $M_\pm=0$, 2 are the special cases of instanton with a^2 (or b)$\to\infty$ and 0, respectively.

The gauge field strength and its dual field for the meron solutions are

$$G_{\mu\nu}=2\phi_2(1-ex^2\phi_1)x^2\varepsilon_{\mu\nu\sigma\lambda}x_\rho I_{\rho\sigma}x_\lambda$$
$$+[-2\phi_1+ex^2(\phi_1^2+\phi_2^2)][I_{\mu\nu}+x^2(x_\mu I_{\nu\rho}-x_\nu I_{\mu\rho})x_\rho], \tag{6a}$$
$$^*G_{\mu\nu}=2\phi_2(1-ex^2\phi_1)x^2(x_\mu I_{\nu\rho}-x_\nu I_{\mu\rho})x_\rho$$
$$+[2\phi_1-ex^2(\phi_1^2+\phi_2^2)][(2x)^{-2}(x_\mu\varepsilon_{\nu\rho\sigma\lambda}-x_\nu\varepsilon_{\mu\rho\sigma\lambda})I_{\rho\sigma}x_\lambda]. \tag{6b}$$

The representation $\mathscr{D}$ of SO(4) may be, generally, a reducible one which can be expressed as the direct sum or product of the irreducible representations $\mathscr{D}^{jk}$ (SO(4))†. Corresponding to every irreducible one $\mathscr{D}^{jk}$, a solution (4)–(6) (instanton or meron) is obtained, then, the general form of the SO(4) imbedding instanton and meron solution can be calculated like the way by which the generators of the direct sum or product of the representations are calculated from those of the constituent representations. We discuss here only the properties of the instanton and meron solutions corresponding to the irreducible representation $\mathscr{D}^{jk}$(SO(4)). The properties of the general solutions can be obtained easily.

The generator $I^{jk}_{\mu\nu}$ of $\mathscr{D}^{jk}$(SO(4)) can be expressed as follows

$$\begin{aligned} &I^{jk}_{ab}=\varepsilon_{abc}(L^{jk}_c+K^{3k}_c), && I^{jk}_{a4}=L^{jk}_a-K^{jk}_a,\\ &L^{jk}_a=I^j_a\times\mathbb{1}_{2k+1}, && K^{jk}_a=\mathbb{1}_{2j+1}\times I^k_a, \end{aligned} \tag{7}$$

where I^j_a denotes the generator of the representation $\mathscr{D}^j$(SU(2)) and $\mathbb{1}_n$ is $n\times n$ unit matrix. Thus, we have

$$\mathrm{Tr}(G^{jk}_{\mu\lambda}G^{jk}_{\nu\lambda})=\tfrac{2}{3}C^{jk}_1(\delta_{\mu\nu}-x^{-2}x_\mu x_\nu)[2(\phi_1+\phi_2)-ex^2(\phi_1+\phi_2)^2]^2$$
$$+\tfrac{2}{3}C^{jk}_2(\delta_{\mu\nu}-x^{-2}x_\mu x_\nu)[2(\phi_1-\phi_2)-ex^2(\phi_1-\phi_2)^2]^2, \tag{8}$$

where C^{jk}_1 and C^{jk}_2 are the Casimir operators of SO(4)

$$C^{jk}_1=\mathrm{Tr}(\boldsymbol{L}^2)=j(j+1)(2j+1)(2k+1),$$
$$C^{jk}_2=\mathrm{Tr}(\boldsymbol{K}^2)=k(k+1)(2j+1)(2k+1). \tag{9}$$

For comparison, we write down that term for the instanton solutions

$$\mathrm{Tr}(G^{jk}_{\mu\lambda}G^{jk}_{\nu\lambda})=\delta_{\mu\nu}\left(C^{jk}_1\frac{16a^4}{e^2(x^2+a^2)^4}+C^{jk}_2\frac{16b^4}{e^1(x^2+b^2)^4}\right). \tag{10}$$

The energy–momentum tensor of the meron solutions which has the form as expected from the conformal covariance theories (Actor 1979, Barut and Xu 1982) is non-zero

$$\theta^{jk}_{\mu\nu}=\tfrac{1}{2}\mathrm{Tr}[4G^{jk}_{\mu\lambda}G^{jk}_{\nu\lambda}-\delta_{\mu\nu}G^{jk}_{\rho\lambda}G^{jk}_{\rho\lambda}]$$
$$=\tfrac{1}{3}(\delta_{\mu\nu}-4x^{-2}x_\mu x_\nu)\{C^{jk}_1[2(\phi_1+\phi_2)-ex^2(\phi_1+\phi_2)^2]^2$$
$$+C^{jk}_2[2(\phi_1-\phi_2)-ex^2(\phi_1-\phi_2)^2]^2\}$$
$$=(3e^2x^6)^{-1}(x^2\delta_{\mu\nu}-4x_\mu x_\nu)[C^{jk}_1M^2_+(2-M_+)^2+C^{jk}_2M^2_-(2-M_-)^2], \tag{11}$$

† Obviously, the direct product of two irreducible representations is equivalent to the direct sum of some irreducible ones. However, if we introduce the different solutions for the two irreducible ones, the solution for the direct product of representations is different from the solutions for the direct sum of representations.

but the total energy iz zero

$$E = \int \theta_{44}(\mathrm{d}^3x) = 0. \tag{12}$$

The action for the instanton solutions is

$$\mathscr{I}^{jk} = \tfrac{1}{2}\int (\mathrm{d}^4x)\,\mathrm{Tr}\, G^{jk}_{\mu\nu}G^{jk}_{\mu\nu}$$

$$= \frac{16\pi^2}{3e^2}\begin{cases} C_1^{jk}+C_2^{jk} & \\ C_1^{jk} & \text{when} \\ C_2^{jk} & \end{cases} \begin{matrix} \phi_1+\phi_2\neq 0, & \phi_1-\phi_2\neq 0 \\ \phi_1+\phi_2\neq 0, & \phi_1-\phi_2=0 \\ \phi_1+\phi_2=0, & \phi_1-\phi_2\neq 0. \end{matrix} \tag{13}$$

However, the action for the meron solutions is divergent loarithmically or vanishing ($M_\pm = 0, 2$).

Now, we are going to discuss the topological property of the instanton and meron solutions, it is easy to prove the following identities

$$\begin{aligned} &I^{jk}_{\mu\nu}x_\nu + \tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}I^{jk}_{\nu\rho}x_\lambda = \mathrm{i}x^2 g_1(x)^{-1}\partial_\mu g_1(x),\\ &I^{jk}_{\mu\nu}x_\nu - \tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}I^{jk}_{\nu\rho}x_\lambda = \mathrm{i}x^2 g_2(x)^{-1}\partial_\mu g_2(x),\\ &g_1(x) = \mathscr{D}^j(\hat{\boldsymbol{r}}, -2\psi)\times \mathbb{1}_{2k+1}, \qquad g_2(x) = \mathbb{1}_{2j+1}\times\mathscr{D}^k(\hat{\boldsymbol{r}}, 2\psi) \end{aligned} \tag{14}$$

where $\mathscr{D}^j$ and $\mathscr{D}^k$ are the representations of SU(2), and

$$\begin{aligned} &\hat{\boldsymbol{r}} = \sin\theta\cos\varphi\,\hat{\boldsymbol{x}}_1 + \sin\theta\sin\varphi\,\hat{\boldsymbol{x}}_2 + \cos\theta\,\hat{\boldsymbol{x}}_3 \\ &x_r = \sqrt{x^2}\cos\psi, \qquad \sqrt{\boldsymbol{x}^2} = r = \sqrt{x^2}\sin\psi. \end{aligned} \tag{15}$$

Therefore, W_μ can be expressed as follows

$$W^{jk}_\mu = \tfrac{1}{2}\mathrm{i}(\phi_1+\phi_2)x^2 g_1^{-1}\partial_\mu g_1 + \tfrac{1}{2}\mathrm{i}(\phi_1-\phi_2)x^2 g_2^{-1}\partial_\mu g_2, \tag{16}$$

and has the explicit form

$$W^{jk}_\mu = \frac{x^2}{x^2+a^2}\frac{\mathrm{i}}{e}g_1^{-1}\partial_\mu g_1 + \frac{x^2}{x^2+b^2}\frac{\mathrm{i}}{e}g_2^{-1}\partial_\mu g_2, \tag{17}$$

for the instanton solutions, and

$$W^{jk}_\mu = (\mathrm{i}/2l)M_+ g_1^{-1}\partial_\mu g_1 + (\mathrm{i}/2l)M_- g_2^{-1}\partial_\mu g_2 \tag{18}$$

for the meron solutions.

Their topological charge is

$$\begin{aligned} g^{jk} &= \frac{e^2}{16\pi^2}\int_{x^2\to\infty}\mathrm{d}S_\mu\varepsilon_{\mu\nu\rho\lambda}\mathrm{Tr}\{W^{jk}_\nu[\partial_\rho W^{jk}_\lambda - \partial_\lambda W^{jk}_\rho - \mathrm{i}e\tfrac{2}{3}(W^{jk}_\rho W^{jk}_\lambda - W^{jk}_\lambda W^{jk}_\rho)]\}\\ &= \frac{e^2}{4\pi^2}\int_{x^2\to\infty}\mathrm{d}\Omega\, x^4\{C_1^{jk}[(\phi_1+\phi_2)^2 - \tfrac{1}{3}ex^2(\phi_1+\phi_2)^3]\\ &\quad + C_2^{jk}[-(\phi_1-\phi_2)^2 + \tfrac{1}{3}ex^2(\phi_1-\phi_2)^3]\}. \end{aligned} \tag{19}$$

For the instanton solutions

$$g^{jk}=\begin{cases}\frac{2}{3}(2j+1)(2k+1)[j(j+1)-k(k+1)] & \phi_1+\phi_2\neq 0, \quad \phi_1-\phi_2\neq 0\\ \frac{2}{3}(2j+1)(2k+1)j(j+1) & \text{when}\quad \phi_1+\phi_2\neq 0, \quad \phi_1-\phi_2=0\\ -\frac{2}{3}(2j+1)(2k+1)k(k+1) & \phi_1+\phi_2=0, \quad \phi_1-\phi_2\neq 0,\end{cases} \tag{20}$$

and for the meron solutions

$$g^{jk}=\tfrac{1}{3}(2j+1)(2k+1)[M_+j(j+1)-M_-k(k+1)]. \tag{21}$$

The imbedding SO(4) meron and instanton solutions and their topological charge are listed in table 1.

Table 1. The imbedding SO(4) meron and instanton solutions and their topological charge.

$\phi_1+\phi_2$	$\frac{1}{ex^2}$	0	$\frac{1}{ex^2}$	$\frac{2}{e(x^2+a^2)}$	$\frac{1}{ex^2}$	$\frac{2}{e(x^2+a^2)}$	0	$\frac{2}{e(x^2+a^2)}$
$\phi_1-\phi_2$	0	$\frac{1}{ex^2}$	$\frac{1}{ex^2}$	$\frac{1}{ex^2}$	$\frac{2}{e(x^2+b^2)}$	0	$\frac{2}{e(x^2+b^2)}$	$\frac{2}{e(x^2+b^2)}$
g_{jk}	$\frac{1}{3}C_1^{jk}$	$-\frac{1}{3}C_2^{jk}$	$\frac{1}{3}(C_1^{jk}-C_2^{jk})$	$\frac{1}{3}(2C_1^{jk}-C_2^{jk})$	$\frac{1}{3}(C_1^{jk}-2C_2^{jk})$	$\frac{2}{3}C_1^{jk}$	$-\frac{2}{3}C_2^{jk}$	$\frac{2}{3}(C_1^{jk}-C_2^{jk})$

Through the gauge transformation

$$U=\mathscr{D}^j(\hat{\boldsymbol{r}},2\alpha_1)\times\mathscr{D}^k(\hat{\boldsymbol{r}},-2\alpha_2)$$
$$\alpha_1=\frac{r}{\sqrt{r^2+a^2}}\left(\tan^{-1}\frac{x_4}{\sqrt{r^2+a^2}}+\frac{\pi}{2}\right),\qquad \alpha_2=\frac{r}{\sqrt{r^2+b^2}}\left(\tan^{-1}\frac{x_4}{\sqrt{r^2+b^2}}+\frac{\pi}{2}\right) \tag{22}$$

the instanton solution turns to

$$W_4^{jk}=0,\qquad \boldsymbol{W}^{jk}=\frac{\mathrm{i}}{e}u^{-1}\boldsymbol{\nabla}u,\qquad u=\mathscr{D}^j\left(\hat{\boldsymbol{r}},-\frac{2\pi r}{\sqrt{r^2+a^2}}\right)\times\mathscr{D}^k\left(\hat{\boldsymbol{r}},\frac{2\pi r}{\sqrt{r^2+b^2}}\right) \tag{23a}$$

for the distant future ($x_4\to\infty$) and

$$W_4^{jk}=0,\qquad \boldsymbol{W}^{jk}=0,\qquad x_4\to-\infty \tag{23b}$$

for the distant past. Therefore, as pointed out by Jackiw (1977), the instanton solution is the evidence for the existence of quantum tunnelling between two classically allowed vacuum configurations.

Through the gauge transformation

$$U=\mathscr{D}^j(\hat{\boldsymbol{r}},-\psi)\times\mathscr{D}^k(\hat{\boldsymbol{r}},\psi), \tag{24}$$

the meron solution becomes

$$W_4^{jk}=0,\qquad \boldsymbol{W}^{jk}=2\sin^2\tfrac{1}{2}\psi(\hat{\boldsymbol{r}}/er)\wedge[M+\boldsymbol{I}^j+\mathbb{1}_{2k+1}+M_-\mathbb{1}_{2j+1}\times\boldsymbol{I}^k]. \tag{25}$$

We can see that the meron solution with $M_+=M_-=1$ becomes the trivial vacuum

$$W_4^{jk}=0,\qquad \boldsymbol{W}^{jk}=0,\qquad \text{when } x_4\to\infty \tag{26a}$$

for the distant future, and to that equivalent to vacuum

$$W_4^{jk}=0, \qquad \boldsymbol{W}^{jk}=\frac{\mathrm{i}}{e}u^{-1}\nabla u, \qquad u=\mathscr{D}^j(\hat{\varphi},2\theta)\times\mathscr{D}^k(\hat{\varphi},2\theta) \qquad \text{when } x_4\to-\infty \tag{26b}$$

for the distant past. At $x_4=0$

$$W_4^{jk}=0, \qquad \boldsymbol{W}^{jk}=(\hat{\boldsymbol{r}}/er)\wedge[\boldsymbol{I}^j+\mathbb{1}_{2k+1}+\mathbb{1}_{2j+1}\times\boldsymbol{I}^k], \tag{26c}$$

it is exactly the imbedding Wu–Yang monopole solution. Thus, meron solutions, like those in the SU(2) gauge field, can be interpreted as tunnelling solutions between two vacuum configurations (Actor 1979). The properties of SU(2) meron solution can be generalised for the imbedding SO(4) meron solutions.

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